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Note

On multicolour noncomplete Ramsey graphs of star graphs

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Abstract

Given graphs $\mathrm{G}, \mathrm{G}_1, \ldots, \mathrm{G}_k$, where $k \geq 2$, the notation

$$\mathrm{G} \to (\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_k)$$

denotes that every factorization $\mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \cdots \oplus \mathrm{F}_k$ of G implies $\mathrm{G}_i \subseteq \mathrm{F}_i$ for *at least* one i, $1 \leq i \leq k$. We characterize G for which

$$\mathrm{G} \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$$

and derive some consequences from this. In particular, this gives the value of the graph Ramsey number $\mathcal{R}(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$.

Keywords: Factorization; r-factor; r-factorable; Edge chromatic number; k-regularization

1. Introduction

Noncomplete Ramsey Theory concerns itself with the factorization of noncomplete graphs. More specifically, given graphs $\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_k$, where $k \geq 2$, we use the standard notation

$$\mathrm{G} \to (\mathrm{G}_1, \mathrm{G}_2, \ldots, \mathrm{G}_k) \tag{1}$$

to mean that for every factorization

$$\mathrm{G} = \mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \cdots \oplus \mathrm{F}_k, \tag{2}$$

we have $\mathrm{G}_i \subseteq \mathrm{F}_i$ for *at least* one i, $1 \leq i \leq k$. Recall that a factor F is a spanning subgraph of G, and (2) means that the edges of each factor F_i partition the edges of G. Equivalently, if we k-colour the edges of G, the edges of F_i form

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the k colour classes. The natural problem in noncomplete Ramsey Theory is to determine all such G for which (1) holds. Given that it is often difficult to achieve this, it is often the case that one looks at various necessary conditions satisfied by such G, such as giving bounds on its order, size, minimum or maximum degree, chromatic number and clique number. A related problem is the determination of all G for which (1) holds but such that

$$G \setminus e \not\to (G_1, G_2, \ldots, G_k)$$

for all $e \in E(G)$. Such graphs G are called $(G_1, G_2, \ldots, G_k)$-minimal.

Since the general multicolour problem appears quite difficult even for complete graphs, most work has centered around the case $k = 2$. In what follows, we use the standard notation $\mathcal{K}_n$ for the complete graph of order n, and $\mathcal{K}(m, n)$ for the complete bipartite graph with partite sets of orders m, n. Various necessary conditions for the case $G \to (\mathcal{K}_m, \mathcal{K}_n)$ are known [2,5,6], as is a characterization for G for which $G \to (\mathcal{K}(1, n), \mathcal{K}(1, n))$. The purpose of this article is to explore those graphs G for which

$$G \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)). \tag{3}$$

In view of the fact that

$$G \to (G_1, G_2, \ldots, G_k) \quad \text{if and only if} \quad C \to (G_1, G_2, \ldots, G_k)$$

for some component C of G, provided each G_i is connected, we restrict our attention to connected graphs G.

2. Preliminaries

Since factorization plays a crucial part in the investigation of graphs G for which (1) holds, we first recall some key results. For proofs, we refer the reader to standard texts like [1,11]. We recall that a k-factor of a graph is a k-regular spanning subgraph.

Theorem T1 (*Tutte [10]*). *A nontrivial graph* G *has a* 1*-factor if and only if for every proper subset* S *of vertices of* G*, the number of odd components of* $G \setminus S$ *does not exceed* $|S|$.

Theorem P (*Petersen [8]*). *A nonempty graph* G *is* 2*-factorable if and only if* G *is* $2n$*-regular for some* $n \geq 1$.

An immediate and useful consequence is that a $2n$-regular graph has a $2m$-factor for each $m < n$. Another useful result is the following.

Lemma A. *For every* $n \geq 1$, $\mathcal{K}_{2n}$ *is* 1*-factorable.*

The concept of a k-factor has a generalization in the following sense. Given a graph G and a function $f : V(G) \to \mathbb{N} \cup \{0\}$, G is said to have an f-factor provided it has a subgraph H such that $\deg_H v = f(v)$ for each $v \in V(G)$. Tutte [9] gave a necessary and sufficient condition for a graph to have an f-factor, relating it to checking whether a related graph (G, f) has a 1-factor. The construction of (G, f) is as follows:

Corresponding to each vertex v of G are complete bigraphs $\mathcal{K}(d(v), e(v))$, with partite sets $A(v)$ of size $d(v) = \deg v$ and $B(v)$ of size $e(v) = \deg v - f(v)$. Corresponding to each edge uv of G, join one vertex of $A(u)$ with one vertex of $A(v)$.

Theorem T2 (*Tutte [9]*). *A graph* G *has an* f*-factor if and only if the graph* (G, f) *has a* 1*-factor.*

The following result, due to U.S.R. Murty, is closely connected and central to our paper.

Theorem M (*Murty*). *Let* G *be a connected graph and n a positive integer. Then* $G \to (\mathcal{K}(1, n), \mathcal{K}(1, n))$ *if and only if*

(a) $\Delta(G) \geq 2n - 1$, *or*
(b) n *is even and* G *is a* $(2n - 2)$*-regular graph of odd order.*

One class of graphs for which the Ramsey numbers are exactly known is the set of graphs each of which is a star graph. Given graphs $G_1, G_2, \ldots, G_k$, where $k \geq 2$, the graph Ramsey number $\mathcal{R}(G_1, G_2, \ldots, G_k)$ is the *least* positive integer p such that $\mathcal{K}_p \rightarrow (G_1, G_2, \ldots, G_k)$. Graph Ramsey numbers generalize the notion of Ramsey numbers $\mathcal{R}(n_1, n_2, \ldots, n_k)$, where $n_1, n_2, \ldots, n_k$ are positive integers:

$$\mathcal{R}(n_1, n_2, \ldots, n_k) := \mathcal{R}\left(\mathcal{K}_{n_1}, \mathcal{K}_{n_2}, \ldots, \mathcal{K}_{n_k}\right).$$

Theorem BR (*Burr and Roberts [3]*). *Let $n_1, n_2, \ldots, n_k$ be positive integers, e of which are even. Then*

$$\mathcal{R}(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)) = \begin{cases} N+1 & \text{if } e \text{ is even and positive;} \\ N+2 & \text{otherwise,} \end{cases}$$

where $N = \sum_{i=1}^{k}(n_i - 1)$.

One of the most fundamental results on edge colouring was proven by Vizing [12], and later independently, by Gupta [7].

Theorem V (*Vizing [12]; Gupta [7]*). *For any simple graph G with maximum vertex degree Δ, the edge chromatic number, $\chi'(G)$ satisfies the inequality*

$$\Delta \le \chi'(G) \le 1 + \Delta.$$

3. Main results

We shall assume throughout that $n_1, n_2, \ldots, n_k$ are arbitrary positive integers and that $N = (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)$. We shall denote the condition

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)) \tag{4}$$

by stating that G satisfies $(n_1, n_2, \ldots, n_k)$. We begin by giving some simple necessary conditions on graphs G which satisfy $(n_1, n_2, \ldots, n_k)$.

Lemma 1. *Let G be a connected graph with p vertices and q edges. If*

$$G \rightarrow (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)),$$

then

(a) $p \ge R(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$, *and*
(b) $q \ge N + 1$.

Moreover, the bounds are sharp.

Proof. Suppose G is a connected graph which satisfies $(n_1, n_2, \ldots, n_k)$.

(a) If $p < R = \mathcal{R}(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$, by definition of the Ramsey number R, $\mathcal{K}_p$ would not satisfy $(n_1, n_2, \ldots, n_k)$. But then nor would G since G has the same order as $\mathcal{K}_p$.
To show this bound is best possible, consider the complete graph $\mathcal{K}_R$. By the definition of a Ramsey number, this satisfies $(n_1, n_2, \ldots, n_k)$, and clearly has order R.
(b) If $q \le N$, G could be factorized into k factors, with $n_i - 1$ edges in each factor for $1 \le i \le k$. But then G does not satisfy $(n_1, n_2, \ldots, n_k)$.
The star graph $\mathcal{K}(1, N+1)$ satisfies $(n_1, n_2, \ldots, n_k)$ and has size $N + 1$, so that the bound for q is best possible. □

We need a construction before our next result. Given a graph G, we may construct a $\Delta(G)$-regular graph $G^\star$ of which G is a induced subgraph. If G is not regular, we make two copies of G and join identical vertices whose degree is not maximal. This results in a graph in which the difference between Δ and δ has decreased by 1. Repetition of this process $\Delta(G) - \delta(G)$ times provides the graph $G^\star$. We call $G^\star$ the Δ-regularization of G. This construction is apparently due to D. König (see [4], p. 40). More generally, for each $k \ge \Delta$, this process can now be extended by increasing the degree of each vertex to arrive at a k-regular supergraph of G, which we denote by $G_k^\star$ and call its k-regularization. There is a simple connection between a graph and its regularization in a specific instance as the following result shows.

Lemma 2. *Let $n_1, n_2, \ldots, n_k$ be positive integers. If $\Delta(\mathrm{G}) = N$, then*

$$\mathrm{G} \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$$

if and only if

$$\mathrm{G}^\star \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)).$$

Proof. Suppose $\mathrm{G}^\star$ satisfies $(n_1, n_2, \ldots, n_k)$ and G does not. Then $\mathrm{G} = \mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \cdots \oplus \mathrm{F}_k$, with $\Delta(\mathrm{F}_i) \le n_i - 1$ for $1 \le i \le k$. However, $\Delta(\mathrm{G}) = N$ forces $\Delta(\mathrm{F}_i) = n_i - 1$ for $1 \le i \le k$. The edge sum of the Δ-regularization of these factors is then N-regular, and hence it is the Δ-regularization of G. But this contradicts our assumption that $\mathrm{G}^\star$ satisfies $(n_1, n_2, \ldots, n_k)$. The converse is trivial. □

Lemma 3. *If* G *is r-regular, then $\mathrm{G}^\star_{r+1}$ is* 1*-factorable.*

Proof. Let G be an r-regular graph. Then, by Theorem V, G is $r + 1$ edge-colourable. The construction of $\mathrm{G}^\star_{r+1}$ involves making two copies of G and joining identical vertices. We use identical colours for edges in the two copies of G, using $r + 1$ colours. Moreover, since each vertex v has degree r in G, only r colours are used for colouring the edges incident with v in each copy. Thus, there is a colour free for the edge joining identical vertices in the two copies. This proves that $\mathrm{G}^\star_{r+1}$ is also $r + 1$ edge-colourable. Then the spanning subgraphs with edges from each of the $r + 1$ colour classes are 1-factors of $\mathrm{G}^\star_{r+1}$. □

Lemma 4. *Let $n_1, n_2, \ldots, n_k$ be positive integers, and let* G *be a connected graph. The following are equivalent:*

(a) $\mathrm{G} \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)) \implies \Delta(\mathrm{G}) \ge N$ *holds for every choice of positive integers* $n_1, n_2, \ldots, n_k$.
(b) G *is r-regular* $\Longrightarrow \mathrm{G}^\star_{r+1}$ *is* 1*-factorable.*

Proof. (a) ⇒ (b): Let G be an r-regular graph. Then

$$\mathrm{G} \not\to \underbrace{(\mathcal{K}(1, 2), \mathcal{K}(1, 2), \ldots, \mathcal{K}(1, 2))}_{r+1 \text{ terms}},$$

since $\Delta(\mathrm{G}) = r = N - 1$ and part (a) holds. Thus, there is a factorization

$$\mathrm{G} = \mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \cdots \oplus \mathrm{F}_{r+1},$$

with $\Delta(\mathrm{F}_i) = 1$ for each i. Since G is r-regular, corresponding to each vertex v of G, there is a factor $\mathrm{F}_{i(v)}$ such that deg v equals 1 in each factor except in $\mathrm{F}_{i(v)}$ where deg $v = 0$.

Fix i, $1 \le i \le r + 1$. Then the subgraph H_i of $\mathrm{G}^\star_{r+1}$ consisting of two copies of F_i together with the edges joining those identical vertices of degree 0 in F_i is a 1-factor of $\mathrm{G}^\star_{r+1}$. Thus, $\mathrm{G}^\star_{r+1}$ is 1-factorable.

(b) ⇒ (a): Suppose G satisfies $(n_1, n_2, \ldots, n_k)$ and $\Delta(\mathrm{G}) < N$. Since $\mathrm{G}^\star_{N-1}$ is an $(N - 1)$-regular graph, by part (b), $\mathrm{G}^\star_N$ is 1-factorable. Thus, we can write

$$\begin{aligned} \mathrm{G}^\star_N &= \mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \cdots \oplus \mathrm{F}_N \\ &= \mathrm{H}_1 \oplus \mathrm{H}_2 \oplus \cdots \oplus \mathrm{H}_k, \end{aligned}$$

where each factor F_i is 1-regular, H_1 equals $\mathrm{F}_1 \oplus \cdots \oplus \mathrm{F}_{n_1-1}$, H_2 equals the edge sum of the next $(n_2 - 1)\mathrm{F}_i$'s, and so on, so that each factor H_i is $(n_i - 1)$-regular. But then

$$\mathrm{G} = (\mathrm{H}_1 \cap \mathrm{G}) \oplus (\mathrm{H}_2 \cap \mathrm{G}) \oplus \cdots \oplus (\mathrm{H}_k \cap \mathrm{G})$$

implies that G does not satisfy $(n_1, n_2, \ldots, n_k)$. This contradiction proves $\Delta(\mathrm{G}) \ge N$. □

Theorem 1. *Let* G *be a connected graph such that*

$$\mathrm{G} \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)).$$

Then $\Delta(\mathrm{G}) \ge N$.

Proof. This is a direct consequence of Lemmas 3 and 4. □

Theorem 2. *Let* G *be a connected graph, and let* $n_1, n_2, \ldots, n_k$ *be positive integers of which e are even. Let* $G^\star$ *be the* Δ*-regularization of* G*, as defined above. Then*

$$G \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$$

if and only if

(a) $\Delta(G) \geq N + 1$, *or*
(b) G *is* N*-regular, of odd order and e is even and non-zero, or*
(c) G *is* N*-regular, of even order, at least one* n_i *is even, and* G *does not have an* $(n_i - 1)$*-factor for at least one even* n_i*, or*
(d) G *is not* N*-regular,* $\Delta(G) = N$ *and* $G^\star \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$.

Proof. Suppose first that *at least* one of the conditions is met.

Condition (a) implies G satisfies $(n_1, n_2, \ldots, n_k)$ by the Pigeonhole Principle. Suppose condition (b) holds and $G = F_1 \oplus F_2 \oplus \cdots \oplus F_k$, with $\Delta(F_i) \leq n_i - 1$ for $1 \leq i \leq k$. The regularity of G forces each F_i to be $(n_i - 1)$-regular. But each F_i is of odd order and at least one $n_i - 1$ is odd, and this is a contradiction.

Suppose next that condition (c) is met. Arguing as in the previous case, we observe that if G did not satisfy $(n_1, n_2, \ldots, n_k)$, it must have an $(n_i - 1)$-factor for each i. But G does not have such a factor for even n_i, and so this is not the case.

Finally, suppose condition (d) holds. By Lemma 2, G satisfies $(n_1, n_2, \ldots, n_k)$. This completes the sufficiency of each of the four conditions.

Conversely, suppose G satisfies $(n_1, n_2, \ldots, n_k)$. If $\Delta(G) \geq N + 1$, there is nothing to prove; suppose $\Delta(G) \leq N$.

If G is N-regular and of odd order, then N must be even, so that e must be even. If $e = 0$, each $n_i - 1$ is even and since G is 2-factorable by Theorem P, G has an $(n_i - 1)$-factor for each i. This contradicts the assumption that G satisfies $(n_1, n_2, \ldots, n_k)$. Thus, in this case, e must be non-zero.

Suppose that G is N-regular and of even order. If G has an $(n_i - 1)$-factor for each even n_i, then the graph obtained from G by removing each of these factors is regular of even degree, and hence 2-factorable, and so has $(n_i - 1)$-factors for odd n_i as well. But then G does not satisfy $(n_1, n_2, \ldots, n_k)$, and this contradiction implies that G does not have $(n_i - 1)$-factors for at least one even n_i.

Finally, suppose G is not N-regular. Since G satisfies $(n_1, n_2, \ldots, n_k)$, $\Delta(G) = N$ by Theorem 1, and clearly $G^\star$ satisfies $(n_1, n_2, \ldots, n_k)$ as well. This completes the characterization. □

The characterization of G that satisfies $(n_1, n_2, \ldots, n_k)$ as given by Theorem 2 makes it easy to determine the Ramsey numbers and the bipartite Ramsey numbers of star graphs. Ramsey numbers of star graphs were determined by Burr and Roberts (Theorem BR in Section 2). However, our proof derives their result as a consequence of a more general result, and is not restricted to determining only complete graphs that satisfies $(n_1, n_2, \ldots, n_k)$.

Corollary 1 (*Burr and Roberts [3]*). *Let* $n_1, n_2, \ldots, n_k$ *be positive integers, e of which are even. Then*

$$\mathcal{R}(\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k)) = \begin{cases} N + 1 & \text{if } e \text{ is even and positive;} \\ N + 2 & \text{otherwise,} \end{cases}$$

where $N = \sum_{i=1}^{k} (n_i - 1)$.

Proof. This is a direct consequence of Theorem 2. Observe that $\mathcal{K}_{N+2}$ satisfies $(n_1, n_2, \ldots, n_k)$ by condition (a). To complete the proof, we need to show that $\mathcal{K}_{N+1}$ satisfies $(n_1, n_2, \ldots, n_k)$ if and only if e even and non-zero. If e is even and non-zero, condition (b) applies to $\mathcal{K}_{N+1}$. Conversely, suppose $\mathcal{K}_{N+1}$ satisfies $(n_1, n_2, \ldots, n_k)$. If N is even, by condition (b), e is even and non-zero. If N is odd, by condition (c), $\mathcal{K}_{N+1}$ does not have an $(n_i - 1)$-factor for at least one even n_i, which contradicts Lemma A. □

Corollary 2. *Let* $n_1, n_2, \ldots, n_k$ *be positive integers, and let* $N = (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1)$. *Then*

$$\mathcal{K}(p, p) \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$$

if and only if $p \geq N + 1$.

Proof. Suppose $\mathcal{K}(p, p) \to (\mathcal{K}(1, n_1), \mathcal{K}(1, n_2), \ldots, \mathcal{K}(1, n_k))$. Since $\mathcal{K}(p, p)$ is 1-factorable, $\mathcal{K}(N, N)$ has an $(n_i - 1)$-factor for each i, $1 \leq i \leq k$, so that $\mathcal{K}(N, N)$ does not satisfy $(n_1, n_2, \ldots, n_k)$. Therefore, $p > N$. Conversely, $\mathcal{K}(N + 1, N + 1)$, and hence $\mathcal{K}(p, p)$ for each $p \geq N + 1$, satisfies $(n_1, n_2, \ldots, n_k)$ because of the Pigeonhole Principle. □

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